

Nonorthogonality analysis of a thermoacoustic system with a premixed V-shaped flame



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ABSTRACT

Thermoacoustic instability occurs in many combustion systems, such as aero-engine afterburners, rocket motors, ramjets and gas turbines. It most often arises due to the coupling between unsteady heat release and acoustic waves. In this work, nonorthogonality analysis of a choked combustor with a gutter confined is conducted. Such configuration is used as a simplified model of the afterburner of an aero-engine. A thermoacoustic model is developed first to study the nonnormal interaction between acoustic disturbances and a premixed V-shaped flame anchored to the tip of the gutter. Eigenmode nonorthogonality analysis is then conducted. The thermoacoustic system is shown to be nonnormal and characterized by nonorthogonal eigenmodes. The nonorthogonality is identified to arise from both the complex boundary condition and the monopole-like flame. However, the contribution from the Robin-type boundary is approximately 1.5% of that from the flame. Thus the flame is identified to play a dominant role. One practical conclusion is that acoustic disturbances undergo transient growth in a combustion system with nonorthogonal eigenmodes. Such finite-time growth, which cannot be predicted by using classical linear theory might trigger high-amplitude self-sustained oscillations.

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1. Introduction

Thermoacoustic instabilities [1–3] are characterized by large-amplitude self-excited pressure oscillations [4–13] caused by a coupling between acoustic waves and unsteady combustion. The acoustic waves perturb the flame and change the instantaneous heat release rate [14,15]. Since the unsteady heat release is an efficient monopole-like sound source [16], the acoustic waves can gain energy from their interaction with unsteady combustion. This feedback can result in the pressure oscillation amplitudes successively increasing [17,18]. Eventually, some nonlinearity in the combustion system will limit the amplitude of oscillations [19]. The combustion-excited oscillations might be so intense as to cause structural damage and costly mission failure [1].

The occurrence of thermoacoustic instabilities has been a plaguing problem in the development of combustors for rocket motors [1], ramjet [20], the afterburner of aero-engines, and land-based gas turbines [1]. Over the last decades, thermoacoustic instabilities have been the subject of intense research activity, aiming to better understand and predict them [21]. Linear stability analysis [22] is generally conducted for studying the stability

behaviors of combustion systems via calculating eigenfrequencies. A combustion system is linearly stable, when all eigenmodes decay exponentially. However, when the eigenmodes are not orthogonal, the combustion system is nonnormal and acoustic disturbances present will undergo transient growth [23–26]. If the transient growth rate is large enough, thermoacoustic instability might be triggered by causing small-amplitude acoustic disturbances to grow to amplitudes high enough to make nonlinear effects significant.

Nonnormality analysis of a solid rocket motor was attempted by Mariappan and Sujith [24]. However, the boundary conditions are modeled as acoustically closed at both ends for simplicity. This means that the rocket motor boundary conditions are Dirichlet-type ones. It was shown that transient growth of acoustic disturbances occurred in such nonnormal system. Similarly, a Rijke-type combustion system with Dirichlet-type boundary conditions was investigated [23,25]. A heat source is modeled and simply described by using time-lag formulation. The interaction between a flame and the acoustic waves was not discussed. This partially motivated the present work.

In this work, nonorthogonality analysis of a longitudinal combustor with a choked inlet and an open exit are conducted. A V-shaped premixed flame anchored on a bluff-body/flame-holder is considered and it is assumed acoustically compact. The flow

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Nomenclature

A_f	flame surface area, m ²	z_u, z_d	axial position of inlet and outlet, m
$\mathbf{A}$	coefficient matrix	$\hat{Z}$	acoustic impedance, m/s
$\mathbf{B}$	coefficient matrix	$\hat{z}_u, \hat{z}_d$	reflection coefficient at inlet and outlet
$\bar{c}_u, \bar{c}_d$	speed of sound upstream and downstream of the flame, m/s	Greek letters	
c_p, c_v	heat capacity ratio at constant pressure and volume, kJ/kg K	χ	coefficient in Eq. (22)
$\mathbf{c}_T$	identity vector	$\delta(z)$	dirac delta function
G	G-equation	ϵ	coefficient in Eq. (15)
h	specific enthalpy, W/m ² K	γ	the ratio of specific heat
H	Heaviside step function	λ_k	weighting factor corresponding to the trapezoidal integration
j	$j = \sqrt{-1}$	ϕ_0	initial phase, rad
M_u, M_d	Mach number upstream and downstream of the flame	ω	oscillation frequency, rad/s
$\mathcal{N}$	interaction index	σ_m	grow rate of m th eigenmode
$\mathbf{n}$	the unit vector of the flame propagation	Σ	Rayleigh index
p'	pressure fluctuation, Pa	ρ	air density, kg/m ³
$\bar{p}$	mean pressure, Pa	$\xi(r, t)$	flame front location
q	heat flux, W/m ²	τ_v	time delay between heat and pressure fluctuation, s
Q	instantaneous heat production per unit volume J/m ³	τ_u, τ_d	time delays, s
$Q', \bar{Q}$	fluctuating and mean heat release rate, kJ/s	Ω_m	eigenvalues
r	radius of the flame front, m	Re	real part
R	gas constant, J/mol K	Im	imaginary part
R_a, R_b	radius of the burner and the tube, m	Subscript	
S_t	flame Strouhal number	u	upstream
T	Temperature, K	$d,$	downstream
T_r	flame transfer function	o	branch outlet
$\mathbf{u}_f$	unburned gas velocity, m/s	f	flame
$\bar{u}_u, \bar{u}_d$	mean velocity upstream and downstream of the flame, m/s	m	m th eigenmode
$\bar{u}_f, u'_f$	mean and fluctuating part of velocity at flame position, m/s	Superscript	
V_f	flame speed, m/s	—	mean value
$\mathbf{V}_N$	state vector in Eq. (41)	$\wedge$	Fourier transform
$\mathbf{W}$	coefficient matrix in Eq. (41)	$*$	complex conjugate
z	axial position, m	$'$	fluctuation part
z_f	heat source location along z , m		

fluctuations are expanded using decomposed traveling plane waves. In Section 2, the thermoacoustic model equations are developed. The flame front is tracked by using the classical G-equation. In Section 3 an analytical method to quantify eigenmodes nonorthogonality is introduced. In Section 4, the flame response to oncoming acoustic disturbances are discussed in terms of the flame transfer function and the time evolution of the flame front. In addition, the contributions to the eigenmodes nonorthogonality from the flame and the choked boundary are compared and discussed. To gain insight on the effect of Robin-type boundary condition on the nonorthogonality, the combustor in the absence of the flame is considered.

2. Thermoacoustic model of the premixed choked combustor

The numerical model considers a confined premixed V-shaped flame burning in the recirculation zone of a bluff body (flame-holder) in a longitudinal combustor. The combustor is associated with a choked inlet and an open exit. Such configuration is one of the simplest generic geometries of an aeroengine afterburner with Robin-type boundary condition [26,27]. It has been extensively investigated experimentally [28,29]. However, nonorthogonality and nonnormality analyses of such combustion system have not been reported yet. For this, a thermoacoustic model with a V-shaped turbulent flame is developed. The numerical model

captures two physical processes; the generation and propagation of the acoustic waves within the combustor, and the dynamic response of the flame to the acoustic waves. Here, the flame front is tracked in real-time by the kinematic G-equation, of which a time-invariant turbulent flame speed is used.

2.1. Model for combustion-driven acoustic waves

A premixed flame with constant fuel-air ratio burning in a cylindrical combustor is considered, as shown in Fig. 1. The geometry of the combustor and flow conditions are summarized in Table A.1 in Appendix A. The flame, stabilized in the wake of an axisymmetric flame-holder (center-body), is acoustically compact

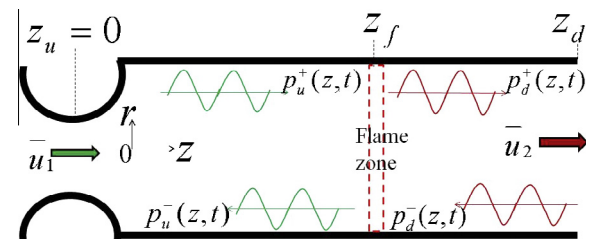


Fig. 1. Schematic of a cylindrical duct with a premixed V-shaped flame anchored on the flame-holder/gutter.

and it is modeled as a thin sheet. The presence of the flame results in the mean temperature $\bar{T}$, the speed of sound $\bar{c}$ and the gas density $\bar{\rho}$ undergoing a sudden jump. Thus, we use subscript u and d to denote the parameters in the pre-burn region and after-flame region respectively. The length of the combustor is L and the flame is being axially placed at z_f . The pressure of the gas mixtures in the combustor is always low enough to justify the assumption of the perfect gas laws. Thus if p denotes the pressure, T the temperature and ρ the density, $p = RT\rho = \rho TR_u/M_w$. Here, M_w is the mean molecular mass of the mixture and R_u is the universal gas constant.

Consider a control volume enclosing the acoustically compact flame, the density ρ varies through changes in both the pressure p and specific enthalpy h . Thus

$$\frac{d\rho}{dt} = \frac{1}{c^2} \frac{dp}{dt} + \frac{\partial \rho}{\partial h} \frac{dh}{dt} \quad (1)$$

For a perfect gas $\rho = p/RT = \gamma p/c^2$, $\partial \rho / \partial h|_p = -\rho(\gamma - 1)/c^2$ and so Eq. (1) can be rewritten as

$$\frac{d\rho}{dt} = \frac{1}{c^2} \frac{dp}{dt} - \frac{\rho(\gamma - 1)}{c^2} \frac{dh}{dt} \quad (2)$$

where γ is the ratio of specific heat. When viscous and heat conduction effects are neglected, $\rho dh/dt = Q$, where $Q(t) = \bar{Q} + Q'(t)$ is the instantaneous heat input/unit volume. An over-bar denotes a mean value and a prime denotes a perturbation.

The mean flow has vanishingly small Mach number (i.e. $M^2 \ll 1.0$), so that the flow properties such as density ρ and pressure p can be assumed to consist of a mean and a fluctuating part as $\rho = \bar{\rho} + \rho'$ and $p = \bar{p} + p'$. The mass and momentum equations over the control volume, when linearized with respect to the perturbation quantities, can be written as

$$\frac{1}{c^2} \frac{\partial p'}{\partial t} + \bar{\rho} \frac{\partial u'}{\partial z} = \frac{(\gamma - 1)}{c^2} Q'(z_f, t) \delta(z - z_f) \quad (3)$$

$$\bar{\rho} \frac{\partial u'}{\partial t} + \frac{\partial p'}{\partial z} = 0 \quad (4)$$

By differentiating Eq. (3) with respect to time t , Eq. (4) with respect to z and subtracting, a general combustion-driven wave equation is obtained namely,

$$\frac{1}{c^2} \frac{\partial^2 p'}{\partial t^2} - \frac{\partial^2 p'}{\partial z^2} = \frac{\gamma - 1}{c^2} \frac{\partial}{\partial t} (Q'(t) \delta(z - z_f)) \quad (5)$$

where $\delta(z)$ is dirac delta function describing localized compact heat source. When there is no heat input, i.e. $Q(t) = 0$, Eq. (5) reduces to the well-known acoustic wave equation:

$$\frac{1}{c^2} \frac{\partial^2 p'}{\partial t^2} - \frac{\partial^2 p'}{\partial z^2} = 0 \quad (6)$$

If the acoustic waves in different combustion regions are modeled as right- and left traveling waves $p^+(z, t)$ and $p^-(z, t)$, as shown in Fig. 1, then the instantaneous pressure fluctuation upstream and downstream of the flame can be described as

$$p(z, t) = \begin{cases} \bar{p} + \left[p_u^+ \left(t - \frac{(z - z_f)}{\bar{c}_u(1 + \bar{M}_u)} \right) + p_u^- \left(t + \frac{(z - z_f)}{\bar{c}_u(1 - \bar{M}_u)} \right) \right] & z_u \leq z \leq z_f \\ \bar{p} + \left[p_d^+ \left(t - \frac{(z - z_f)}{\bar{c}_d(1 + \bar{M}_d)} \right) + p_d^- \left(t + \frac{(z - z_f)}{\bar{c}_d(1 - \bar{M}_d)} \right) \right] & z_f < z \leq z_d \end{cases} \quad (7)$$

The instantaneous velocity fluctuation is given as

$$u(z, t) = \begin{cases} \bar{u}_u + \frac{1}{\rho_u \bar{c}_u} \left[p_u^+ \left(t - \frac{(z - z_f)}{\bar{c}_u(1 + \bar{M}_u)} \right) - p_u^- \left(t + \frac{(z - z_f)}{\bar{c}_u(1 - \bar{M}_u)} \right) \right] & z_u \leq z \leq z_f \\ \bar{u}_d + \frac{1}{\rho_d \bar{c}_d} \left[p_d^+ \left(t - \frac{(z - z_f)}{\bar{c}_d(1 + \bar{M}_d)} \right) - p_d^- \left(t + \frac{(z - z_f)}{\bar{c}_d(1 - \bar{M}_d)} \right) \right] & z_f < z \leq z_d \end{cases} \quad (8)$$

where $\bar{M}_u$ and $\bar{M}_d$ denote the Mach numbers of upstream and downstream of the flame. Manipulating Eqs. (7) and (8) gives rise to the acoustic traveling waves as given as $p^\pm(z, t) = (p'(z, t) \pm \bar{\rho} \bar{c} u'(z, t))/2$.

The combustor boundaries are modeled using pressure reflection coefficients [25]. At the open downstream end, the reflection coefficient $\hat{Z}_d(\omega)$ is used to couple the reflected wave $p_d^-(z_d, t)$ with the incident one $p_d^+(z_d, t)$ as given in Eq. (9a). And at the choked upstream end, $\hat{Z}_u(\omega)$ is used to couple $p_u^+(z_u, t)$ to $p_u^-(z_u, t)$, as given in Eq. (9b).

$$\hat{p}_d^-(z_d, \omega) = \hat{Z}_d(\omega) \hat{p}_d^+ \exp \left(-\frac{2j\omega(z_d - z_f)}{\bar{c}_d(1 - \bar{M}_d^2)} \right) \quad (9a)$$

$$\hat{p}_u^+(-z_u, \omega) = \hat{Z}_u(\omega) \hat{p}_u^- \exp \left(-\frac{2j\omega(z_f - z_u)}{\bar{c}_u(1 - \bar{M}_u^2)} \right) \quad (9b)$$

Since the mass flow rate at the choked inlet is constant, it can be then shown that the reflection coefficient $\hat{Z}_u = 1 - \bar{M}_u/1 + \bar{M}_u$. The downstream reflection coefficient $\hat{Z}_d$ is set to be to -1.0 for simplicity.

2.2. Model for premixed V-shaped flame dynamics

The inhomogeneous combustion-driven acoustic wave equation as given in Eq. (5) can be solved if $Q'(t)$ is known. Following the work [27], the unsteady heat release $Q'(t)$ from the premixed ducted flame imitates the evolution of the flame surface area ratio such that

$$\frac{Q'(t)}{\bar{Q}} \propto \frac{A_f'(t - \tau)}{\bar{A}_f} \quad (10)$$

To capture the dynamics of the V-shaped flame response with oncoming acoustic disturbances, the classical G-equation is used for flame-front tracking [27] to study the nonnormal interaction between the flame and flow disturbances. The main features of the flame model are reproduced here for completeness and integrity. The flow is assumed to be axisymmetric and combustion is assumed to occur on a thin surface whose axial position at radius r is given by $z_f = \xi(r, t)$ as shown in Fig. 1. The flame front corresponding to a particular level $G(z_f, r, t) = 0$ of a scalar field G describes the flame initiation surface, as, $z_f = \xi(r, t)$. The flame surface is assumed to propagate itself in the direction of its normal $\mathbf{n}_f = \nabla G/|\nabla G|$, into the unburnt fluid at a constant burning velocity V_f relative to the unburnt gas. That is to say the surface $G(z_f, r, t) = 0$ moves in the direction of its normal $\mathbf{n}_f$ with speed $\mathbf{u}_f \cdot \mathbf{n}_f - V_f$, where $\mathbf{u}_f = (u_f, v_f)$ is the unburned gas velocity, and $\mathbf{n}_f$ points downstream as shown in Fig. 1. Mathematically this statement is equivalent to $dG/dt = 0$:

$$\frac{dG}{dt} = \frac{\partial G}{\partial t} + (\mathbf{u}_f - V_f \mathbf{n}_f) \cdot \nabla G \quad (11)$$

After substituting $\mathbf{n}_f = \nabla G/|\nabla G|$ and $G(z_f, r, t) = z_f - \xi(r, t)$, the familiar G-equation is obtained, which describes the location of the flame surface [27]:

$$\frac{\partial \xi(r, t)}{\partial t} + v_f \frac{\partial \xi(r, t)}{\partial r} + V_f \left[1 + \left(\frac{\partial \xi(r, t)}{\partial r} \right)^2 \right]^{1/2} = u_f \quad (12)$$

With the assumption that the density change across the surface $G(z_f, r, t) = 0$ is negligible [27], the particle velocity is given by that in the oncoming flow $(u_f, 0)$ and the equation describing the time evolution of $\xi(r, t)$ is shown as

$$\frac{\partial \xi(r, t)}{\partial t} = u_f(z_f, t) - V_f \left[1 + \left(\frac{\partial \xi(r, t)}{\partial r} \right)^2 \right]^{1/2} \quad (13)$$

It is apparent that the flame front evolution is dynamically coupled with the unsteady velocity $u_f(t)$ at the flame-holder, which is related to the oncoming acoustic disturbance velocity $\bar{u}_u$ as

$$u_f(t) = \frac{R_b^2}{(R_b^2 - R_a^2)} \left[\bar{u}_u + \frac{1}{\rho_u c_u} (p_u^+(z_f, t) - p_u^-(z_f, t)) \right] \quad (14)$$

where R_a and R_b are the radius of the flame-holder and the combustor respectively. When $u_f(t) \geq V_f$, the flame is attached at the flame-holder, i.e. $\xi(R_a, t) = 0$. And Eq. (13) can be rearranged to give

$$\frac{\partial \xi(r, t)}{\partial r} = \sqrt{\frac{u_f^2(t)}{V_f^2} - 1} \quad \text{on } r = R_a \quad (15)$$

However, when $u_f(t) < V_f$, the flame no longer remains to attach at the flame-holder and it propagates upstream. Thus the boundary condition on $r = R_a$ is given as $\partial \xi(r, t)/\partial r = 0$, if $u_f(t) < V_f$ or $\xi(R_a, t) < 0$.

In our calculation, $\xi(r, t)$ is evaluated at 300 locations across the combustor radius and a 4th order Runge–Kutta algorithm is used for the numerical time integration.

Once $\xi(r, t)$ is known, the instantaneous flame surface area $A_f(t)$ can be readily calculated by

$$A_f(t) = \int_0^{2\pi} d\theta \int_{R_a}^{R_b} r \left(1 + \left(\frac{\partial \xi(r, t)}{\partial r} \right)^2 \right)^{1/2} dr \quad (16)$$

Since $\xi(r, t) = \bar{\xi}(r) + \xi'(r, t)$, substituting it into Eq. (16) and using binomial expansion scheme leads to

$$\begin{aligned} A_f(t) &= \int_0^{2\pi} d\theta \int_{R_a}^{R_b} r \sqrt{1 + \left(\frac{\partial \bar{\xi}}{\partial r} \right)^2 + 2 \frac{\partial \bar{\xi}}{\partial r} \frac{\partial \xi'}{\partial r} + \left(\frac{\partial \xi'}{\partial r} \right)^2} dr \\ &= \int_{R_a}^{R_b} 2\pi r \sqrt{1 + \left(\frac{\partial \bar{\xi}}{\partial r} \right)^2} dr + \int_{R_a}^{R_b} 2\pi r \frac{\frac{\partial \bar{\xi}}{\partial r} \frac{\partial \xi'}{\partial r}}{\sqrt{1 + \left(\frac{\partial \bar{\xi}}{\partial r} \right)^2}} dr \\ &\quad + \mathcal{O} \left(\left(\frac{\partial \xi'}{\partial r} \right)^2 \right) + \dots \end{aligned} \quad (17)$$

It can be seen from Eq. (17) that the mean and linearized fluctuating values of the flame surface area are given as

$$\bar{A}_f = \int_{R_a}^{R_b} 2\pi r \sqrt{1 + \left(\frac{d\bar{\xi}(r)}{dr} \right)^2} dr \quad (18)$$

$$A'_f(t) = \int_{R_a}^{R_b} 2\pi r \frac{\frac{\partial \bar{\xi}(r, t)}{\partial r} \frac{\partial \xi'(r, t)}{\partial r}}{\sqrt{1 + \left(\frac{\partial \bar{\xi}(r)}{\partial r} \right)^2}} dr \quad (19)$$

The mean value of Eq. (13) leads directly to $\bar{u}_f = V_f \sqrt{1 + d\bar{\xi}(r)/dr}$. Manipulating it leads to $d\bar{\xi}(r)/dr = \sqrt{\bar{u}_f^2 - V_f^2}/V_f$. After integration and application of boundary condition, it can be shown that

$$\bar{\xi}(r) = (r - R_a) \sqrt{\left(\frac{\bar{u}_f}{V_f} \right)^2 - 1} \quad (20)$$

Substituting Eq. (20) into Eqs. (18) and (19) leads to

$$\bar{A}_f = \frac{\pi(R_b^2 - R_a^2) \bar{u}_f}{V_f} \quad (21)$$

$$A'_f(t) = \sqrt{1 - \frac{V_f^2}{\bar{u}_f^2}} \int_{R_a}^{R_b} 2\pi r \frac{d\xi'(r, t)}{dr} dr = \chi \int_{R_a}^{R_b} \xi'(r, t) dr \quad (22)$$

where $\chi = -2\pi \sqrt{1 - V_f^2/\bar{u}_f^2}$. To simplified Eq. (22), integration by parts and the boundary conditions are applied. If the flame front is discretized into K flame elements equal of radial length δr , then the flame surface area fluctuation can be expressed as

$$A'_f(t) = \sum_{k=1}^K A'_{fk} = \chi \sum_{k=1}^K \lambda_k \xi'_k(t) \delta r \quad (23)$$

where λ_k is the weight factors corresponding to the trapezoidal integration formula as given below

$$\lambda_k = \begin{cases} \frac{1}{2} & k = 1 \text{ and } K \\ 1 & k \neq 1 \text{ or } K \end{cases} \quad (24)$$

Linearizing Eq. (13) and taking Fourier transform leads to a first-order ODE for $\hat{\xi}(r)$ as:

$$j\omega \hat{\xi}(r) = \hat{u}_f - \frac{d\hat{\xi}}{dr} \frac{V_f}{\bar{u}_f} \sqrt{\left(\bar{u}_f^2 - V_f^2 \right)} \quad (25)$$

Further simplification reveals that

$$\hat{\xi}(r) = \frac{\hat{u}_f}{j\omega} \left[1 - e^{\frac{-j\omega(r-R_a)}{V_f \sqrt{1 - V_f^2/\bar{u}_f^2}}} \right] \quad (26)$$

Thus the fluctuating flame surface area is described as

$$\begin{aligned} \hat{A}_f(\omega) &= \frac{2\hat{u}_f(\omega)\pi(R_b - R_a)}{V_f} \left[\frac{R_a}{S_t^2(\omega)} (1 - \cos S_t(\omega) + j(\sin S_t(\omega) - S_t(\omega))) \right. \\ &\quad \left. + \frac{R_b}{S_t^2(\omega)} (j(S_t(\omega) \cos S_t(\omega) - \sin S_t(\omega)) - (1 - \cos S_t(\omega) - S_t(\omega) \sin S_t(\omega))) \right] \end{aligned} \quad (27)$$

where the flame Strouhal number S_t is given as

$$S_t(\omega) = \frac{\omega(R_b - R_a)}{V_f \sqrt{1 - V_f^2/\bar{u}_f^2}} \quad (28)$$

The flame transfer function $T_r(\omega)$ [14,27] between the unsteady heat release and the oncoming flow velocity can be shown as

$$T_r(\omega) = \frac{\hat{Q}(\omega)/\bar{Q}}{\hat{u}_f(\omega)/\bar{u}_f} = \frac{\hat{A}_f(\omega)/\bar{A}_f}{\hat{u}_f(\omega)/\bar{u}_f} \quad (29)$$

where $\hat{A}_f(\omega)$ and $\bar{A}_f$ are given in Eqs. (27) and (21). Thus $T_r(S_t)$ can be rewritten as

$$\begin{aligned} T_r(S_t) &= \frac{-2j}{(R_b + R_a)} \left[\frac{R_a}{S_t^2} (S_t - \sin S_t - j(\cos S_t - 1)) \right. \\ &\quad \left. + \frac{R_b}{S_t^2} (\sin S_t - S_t \cos S_t - j(1 - \cos S_t - S_t \sin S_t)) \right] \end{aligned} \quad (30)$$

The characteristics of the flame transfer function $T_r(S_t)$ is discussed later in Section 4.

2.3. Nonlinear self-sustained thermoacoustic oscillations

Across the acoustically compact flame, the acoustic waves either side of the flame and the heat release fluctuation are related by mass, momentum and energy Eqs. (31)–(33). Substituting Eq. (31) into Eq. (33) and linearizing the momentum and energy Eqs. (32) and (33) give the time evolution of the outgoing traveling waves $p_u^-(t)$ and $p_d^+(t)$ generated by the unsteady heat release $Q'(t)$.

$$\frac{\partial}{\partial t} \left[\pi R_b^2 \int_{z_f}^{z_f^+} \rho dz \right] = \pi R_b^2 [\rho u]_{z_f}^{z_f^+} = 0 \quad (31)$$

$$\frac{\partial}{\partial t} \left[\pi R_b^2 \int_{z_f}^{z_f^+} \rho u dz \right] = \pi R_b^2 \left([p]_{z_f}^{z_f^+} + \rho u [u]_{z_f}^{z_f^+} \right) = 0 \quad (32)$$

$$\frac{\partial}{\partial t} \left[\pi R_b^2 \int_{z_f}^{z_f^+} \left(\frac{\rho u^2}{2} + \frac{\gamma p}{\gamma - 1} \right) dz \right] = \pi R_b^2 \left(\frac{\gamma}{\gamma - 1} [pu]_{z_f}^{z_f^+} + \frac{1}{2} \rho u [u^2]_{z_f}^{z_f^+} - Q(t) \right) = 0 \quad (33)$$

By using these flow conservation equations across the flame and applying the end boundary conditions, the acoustic waves present in the combustion system can be solved. The resulting matrix equation is given as

$$\mathbf{A} \begin{pmatrix} p_u^-(t) \\ p_d^+(t) \end{pmatrix} = \mathbf{B} \begin{pmatrix} p_u^-(t - \frac{2(z_f - z_u)}{c_u(1 - \bar{M}_u^2)}) \\ p_d^+(t - \frac{2(z_d - z_f)}{c_d(1 - \bar{M}_d^2)}) \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{Q'(t)}{c_u \pi R_b^2} \end{pmatrix} \mathbf{v} \quad (34)$$

where $\mathbf{A}$ and $\mathbf{B}$ are coefficient matrices as given in Appendix B. Iteratively solving the system equations enable time evolution of the flow disturbances to be calculated, thus providing a platform to gain insights on the onset of thermoacoustic oscillations as discussed later.

Rayleigh index Σ is generally used as a critical indicator of the combustion system stability. It describes the energy exchange rate between unsteady heat release and the pressure oscillations. If $\Sigma > 0$, the acoustical energy in the present system is increased and tends to drive the unstable process into a limit cycle. However, a negative Σ indicates a 'destruction' interaction between unsteady heat release and pressure oscillations and so the combustion oscillations are decaying. Thus it would be interesting to check the Rayleigh index Σ . Following the work [15], we define a normalized Rayleigh index $\Sigma(t)$ as given as

$$\Sigma(t) = \int_{z_u}^{z_d} \delta(z - z_f) dz \int_0^t \times \frac{p'(\omega, \psi) Q'(\omega, \psi) d\psi}{\sqrt{\int_0^{2\pi/\omega} p^2(\omega, \psi) \sqrt{\int_0^{2\pi/\omega} Q^2(\omega, \psi) d\psi}} d\psi \quad (35)$$

It is then used to characterize the heat-to-sound interaction in the present thermoacoustic system.

3. Nonorthogonality analysis of thermoacoustic modes

In order to gain insight on the stability behaviors of the present combustion system, nonorthogonality analysis of the combustion-excited modes is performed [25,26]. The nonorthogonality is characterized by the inner product of two eigenmodes [25]. If the inner product is zero, then the eigenmodes are orthogonal. The combustion system with decaying orthogonal eigenmodes will be stable. However, when the inner product is non-zero, the combustion system is nonnormal and characterized with nonorthogonal eigenmodes. In such nonnormal system, finite-time growth of acoustic disturbances might trigger thermoacoustic instability. Thus it is critical to characterize the combustion system with or without nonorthogonal eigenmodes and to quantify the nonorthogonality.

3.1. Nonorthogonal eigenmodes

Let's first consider the combustion system as described in Table A.1. The flame results in the mean temperature undergoing a jump, i.e. $\bar{T}_d > \bar{T}_u$. It behaves like an efficient monopole source [22] and produces the traveling acoustic waves as described in Eq. (5). Taking Fourier transform to it and taking the complex conjugate yields

$$-\frac{\omega_n^2}{\gamma \bar{p}} \hat{p}_n^* - \frac{\partial}{\partial z} \left(\frac{1}{\bar{\rho}} \frac{\partial \hat{p}_n^*}{\partial z} \right) = \frac{\gamma - 1}{\gamma \bar{p}} \hat{Q}^* \delta(z - z_f) (j\omega_n)^* \quad (36)$$

Manipulating Eq. (36) leads to

$$\hat{p}_n^* = \frac{-\gamma \bar{p}}{\omega_n^2} \left[\frac{\partial}{\partial z} \left(\frac{1}{\bar{\rho}} \frac{\partial \hat{p}_n^*}{\partial z} \right) - j\omega_n \frac{\gamma - 1}{\gamma \bar{p}} \hat{Q}^* \delta(z - z_f) \right] \quad (37)$$

By using Eq. (37), the inner product between the eigenmodes $\hat{p}_m$ and $\hat{p}_n^*$ can be shown as

$$\langle \hat{p}_m, \hat{p}_n^* \rangle = \int_{z_u}^{z_d} \hat{p}_m \hat{p}_n^* dz = \frac{-\gamma \bar{p}}{\omega_n^2} \int_{z_u}^{z_d} \hat{p}_m \frac{\partial}{\partial z} \left(\frac{1}{\bar{\rho}} \frac{\partial \hat{p}_n^*}{\partial z} \right) dz + \frac{j(\gamma - 1)}{\omega_n} \int_{z_u}^{z_d} \hat{p}_m \hat{Q}^* \delta(z - z_f) dz \quad (38)$$

To determine the inner product, unsteady heat release $\hat{Q}(\omega)$ needs to be estimated. If we assume the flame Strouhal number S_L is small in the present system [17], then the classical time-lag $\mathcal{N} - \tau$ formulation can be used to describe the interaction between $Q'(t)$ and oncoming flow [30]. It has been demonstrated in Section 2 that the kinematic flame model can be linearized into the $\mathcal{N} - \tau$ formulation. In addition, this formulation has been widely used in quantitative prediction of combustion instabilities [30]. According to the $\mathcal{N} - \tau$ formulation, the unsteady heat release is given as

$$Q'(t) = \frac{\gamma \bar{p} \mathcal{N} u_f'(t - \tau_v)}{(\gamma - 1)} \quad (39)$$

where $\mathcal{N}$ denotes the interaction index describing the intensity of the interaction, $u_f(t)$ is the oncoming flow velocity at the flame position and τ_v is the time lag between the velocity perturbation and heat release oscillation, which is assumed to be equal to $0.4z_d/\bar{u}_u$. Here z_d is the length of duct downstream of the flame holder. If we approximate $u_f'(t - \tau_v)$ in Eq. (39) to the first order i.e. $u_f'(t) - \tau_v \partial u_f'(t)/\partial t$, taking Fourier transform to both sides of Eq. (39) and substituting $\hat{Q}(\omega)$ into Eq. (38), then the inner product can be shown as

$$\langle \hat{p}_m, \hat{p}_n^* \rangle = \frac{-\gamma \bar{p}}{\omega_n^2} \int_{z_u}^{z_f} \left(\hat{p}_{u,m}^+ e^{\frac{-j\omega_m(z-z_f)}{c_u(1-\bar{M}_u^2)}} + \hat{p}_{u,m}^- e^{\frac{j\omega_m(z-z_f)}{c_u(1-\bar{M}_u^2)}} \right) \frac{\partial}{\partial z} \left(\frac{1}{\bar{\rho}} \frac{\partial \hat{p}_n^*}{\partial z} \right) dz + \frac{-\gamma \bar{p}}{\omega_n^2} \int_{z_f}^{z_d} \left(\hat{p}_{d,m}^+ e^{\frac{-j\omega_m(z-z_f)}{c_d(1-\bar{M}_d^2)}} + \hat{p}_{d,m}^- e^{\frac{j\omega_m(z-z_f)}{c_d(1-\bar{M}_d^2)}} \right) \frac{\partial}{\partial z} \left(\frac{1}{\bar{\rho}} \frac{\partial \hat{p}_n^*}{\partial z} \right) dz + \frac{j\mathcal{N} \bar{c}_u e^{j\omega_n \tau_v} R_b^2}{\omega_n (R_b^2 - R_d^2)} (\hat{p}_{u,m}^+ + \hat{p}_{u,m}^-) (\hat{p}_{u,n}^{*+} - \hat{p}_{u,n}^{*-}) \quad (40)$$

where ω_m and ω_n are the m th and n th modes frequencies. They can be estimated by solving Eq. (34) in frequency domain as given by the resulting block matrix equation as

$$\mathbf{c}^T \underbrace{\begin{pmatrix} \mathbf{W}_1 & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{W}_2 & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{W}_N \end{pmatrix}}_{\mathbf{w}} \begin{pmatrix} \mathbf{V}_1 \\ \mathbf{V}_2 \\ \vdots \\ \mathbf{V}_N \end{pmatrix} = \mathbf{c}^T \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \end{pmatrix} \quad (41)$$

where $\mathbf{V}_m = [\hat{p}_{u,m}^-(\omega_m), \hat{p}_{d,m}^-(\omega_m)]^T$, $\mathbf{c}^T = [\mathbf{1}, \mathbf{1}, \dots, \mathbf{1}]$. Subscript m denotes the m th mode with frequency of ω_m and $\mathbf{W}_m$ is a 2×2 matrix as given as

$$\mathbf{W}_m = \begin{pmatrix} \mathbf{A} - \mathbf{B} \begin{pmatrix} e^{-j\omega_m \tau_u} & \mathbf{0} \\ \mathbf{0} & e^{-j\omega_m \tau_d} \end{pmatrix} & \begin{pmatrix} \mathbf{0} \\ \frac{\mathcal{N} e^{-j\omega_m \tau_v} (1 - \hat{Z}_u e^{-j\omega_m \tau_u})}{\pi (R_b^2 - R_d^2) (\gamma - 1)} \end{pmatrix} \\ \begin{pmatrix} \mathbf{0} \\ \frac{\mathcal{N} e^{-j\omega_m \tau_v} (1 - \hat{Z}_u e^{-j\omega_m \tau_u})}{\pi (R_b^2 - R_d^2) (\gamma - 1)} \end{pmatrix} & \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \end{pmatrix} \end{pmatrix} \quad (42)$$

where the time delay $\tau_u = 2(z_f - z_u)/\bar{c}_u(1 - \bar{M}_u^2)$ and $\tau_d = 2(z_d - z_f)/\bar{c}_d(1 - \bar{M}_d^2)$ and the matrices $\mathbf{A}$ and $\mathbf{B}$ are given in Eq. (34). Theoretically, the acoustic disturbances consist of infinite modes. For simplicity, we assume that N modes are involved in the flow fluctuations, i.e.

$$\hat{p}_u^- = \sum_{m=1}^N \hat{p}_{u,m}^-(\omega_m), \quad \hat{p}_d^+ = \sum_{m=1}^N \hat{p}_{d,m}^+(\omega_m) \quad (43)$$

Eq. (41) has solutions only if the determinant of the matrix $\mathbf{W}_m$ vanishes, i.e.

$$\det\{\mathbf{W}\} = \prod_{m=1}^N \det\{\mathbf{W}_m\} = 0 \quad (44)$$

The root satisfying $\det\{\mathbf{W}_m\} = 0$, i.e.

$$\Omega_m = \omega_m + j\sigma_m \quad (45)$$

is the eigenfrequencies of the present system with the premixed flame confined. Physically, σ_m denotes the grow rate of thermoacoustic oscillation. When $\sigma_m > 0$, the combustion system is unstable. However, if $\sigma_m < 0$, the long-term evolution of eigenmodes are decaying. However, if the eigenmodes are nonorthogonal, they may interact and transient energy growth might occur. ω_m is the thermoacoustic oscillation mode frequency. They can be found by Newton's iteration method [22], as shown later.

3.2. Nonorthogonality arising from Robin boundary condition

The forgoing analysis shows how to quantify the eigenmodes nonorthogonality of the choked combustion system. However, it is interesting to find out the contribution to the nonorthogonality arising from the complex boundary condition. This can be done by considering the combustor in the absence of flame, i.e. assuming $\mathcal{N} \rightarrow 0$. The nonorthogonality is also quantified by estimating the eigenmodes inner product $\langle \hat{p}_m, \hat{p}_n \rangle$. Here $\hat{p}_m$ and $\hat{p}_n$ satisfy the homogeneous wave Eq. (5) in frequency domain as given as

$$\frac{\partial^2 \hat{p}_m}{\partial z^2} = -\frac{\omega_m^2}{\bar{c}^2} \hat{p}_m \quad (46)$$

$$\frac{\partial^2 \hat{p}_n}{\partial z^2} = -\frac{\omega_n^2}{\bar{c}^2} \hat{p}_n \quad (47)$$

By multiplying Eq. (46) with $\omega_n \hat{p}_n / \omega_m$ and Eq. (47) with $\omega_m \hat{p}_m / \omega_n$, subtracting and integrating over the combustor, the inner product $\langle \hat{p}_m, \hat{p}_n \rangle$ is obtained namely

$$\int_{z_u}^{z_d} \left[\frac{\partial^2 \hat{p}_m}{\partial z^2} \hat{p}_n - \frac{\partial^2 \hat{p}_n}{\partial z^2} \hat{p}_m \right] dz = \frac{(\omega_n^2 - \omega_m^2)}{\bar{c}^2} \int_{z_u}^{z_d} \hat{p}_m \hat{p}_n dz \quad (48)$$

Eq. (48) can be simplified to

$$\begin{aligned} & \left[\frac{\partial \hat{p}_m}{\partial z} \hat{p}_n - \frac{\partial \hat{p}_n}{\partial z} \hat{p}_m \right]_{z_u}^{z_d} + \int_{z_u}^{z_d} \left(\frac{\partial \hat{p}_m}{\partial z} \frac{\partial \hat{p}_n}{\partial z} - \frac{\partial \hat{p}_n}{\partial z} \frac{\partial \hat{p}_m}{\partial z} \right) dz \\ & = \frac{(\omega_n^2 - \omega_m^2)}{\bar{c}^2} \int_{z_u}^{z_d} \hat{p}_m \hat{p}_n dz \end{aligned} \quad (49)$$

It is obvious that the first term involving the boundary conditions plays a dominant role in contributing to the inner product $\langle \hat{p}_m, \hat{p}_n \rangle$, while the second term on left hand side of Eq. (49) is zero. Thus the inner product $\langle \hat{p}_m, \hat{p}_n \rangle$ can be shown as

$$\langle \hat{p}_m, \hat{p}_n \rangle = \int_{z_u}^{z_d} \hat{p}_m \hat{p}_n dz = \frac{\bar{c}^2}{(\omega_n^2 - \omega_m^2)} \left[\frac{\partial \hat{p}_m}{\partial z} \hat{p}_n - \frac{\partial \hat{p}_n}{\partial z} \hat{p}_m \right]_{z_u}^{z_d} \quad (50)$$

Eq. (50) can then be used to characterize the Robin-type boundary effect on the system nonorthogonality. The contributions to the nonorthogonality from the flame and the boundary are compared and discussed in the following section.

4. Results and discussion

The flame response to oncoming flow disturbances are studied first by using the model developed in Section 2. For a given forcing $u_f(t)$, Eq. (13) can be numerically integrated in time to determine the evolution of the flame front. To illustrate the dynamic response of the premixed V-shaped flame, an unsteady inlet flow $u_f(t)$ as given as Eq. (51) is applied to the flame. Here $H(t)$ is the Heaviside step function and ϕ_0 is the initial phase.

$$\frac{u_f(t)}{\bar{u}_f} = 1 + (1 + \epsilon)H(t) \sin(\omega t + \phi_0) \quad (51)$$

Fig. 2 shows the time evolution of the flame front. Comparison is then made with the mean flame position, as denoted by the dot line. It is interesting to observe that the flame moves upstream of the flame-holder during part of the oscillations as $2\pi/4 \leq \omega t \leq 6\pi/4$, since the fluctuating part of the oncoming flow velocity is larger than its mean value.

As shown above, the flame dynamics can also be characterized by using the transfer function $T_r(S_r)$. The variation of $T_r(S_r)$ with the

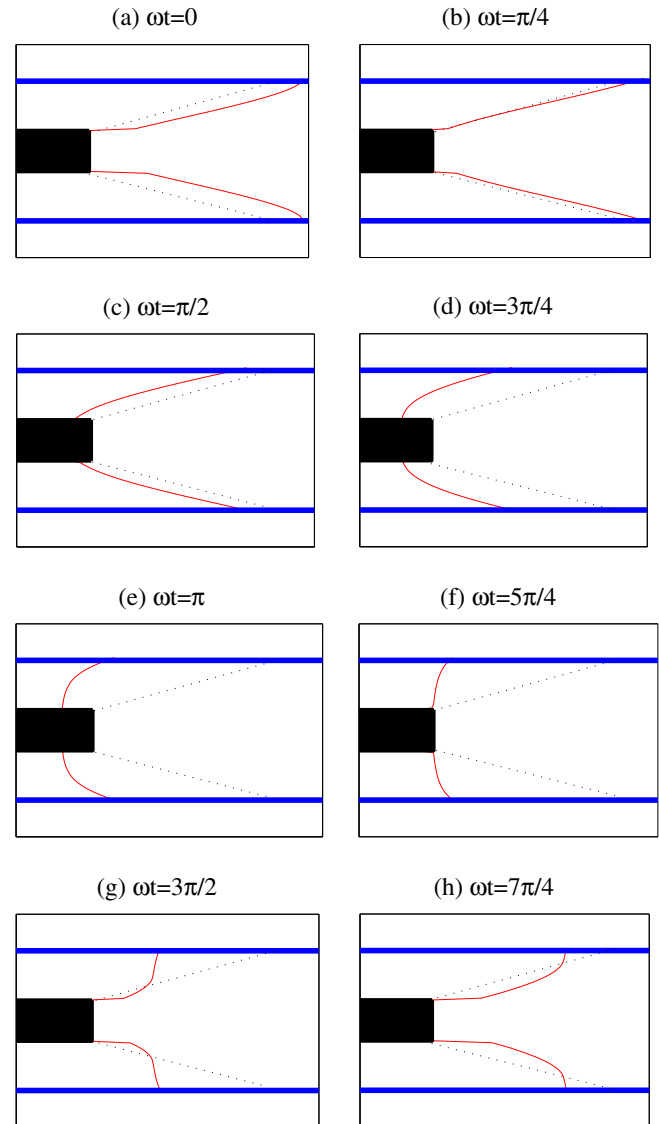


Fig. 2. Time evolution of the premixed V-shaped flame surface anchored to the tip of the gutter during one period of the oncoming acoustic disturbance, as $\epsilon = 0.2$, $R_a/R_b = 0.25$, $\omega/2\pi = 59$ Hz, $V_f = 3.6$ m/s, $\bar{u}_f = 37$ m/s, and $\phi_0 = 1.7$.

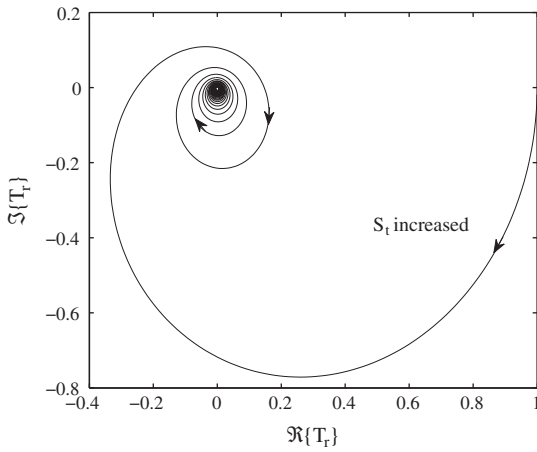


Fig. 3. Variation of flame transfer function T_r with Strouhal number S_t .

flame Strouhal number S_t is shown in Fig. 3. It can be seen that when the flame Strouhal number is negligible (i.e. $S_t \rightarrow 0$ and $\omega \rightarrow 0$), the magnitude of T_r is approximately 1 and the unsteady heat release and oncoming acoustic disturbances velocity is in phase, i.e. $\arctan\{\Im(T_r), \Re(T_r)\} = 0$. However, as S_t is increased, the amplitude of T_r is decreased. And the phase between unsteady heat release and acoustic velocity is increased. Further increasing the flame Strouhal number S_t gives rise to the amplitude of the flame transfer function to reduce to zero. However, the phase is

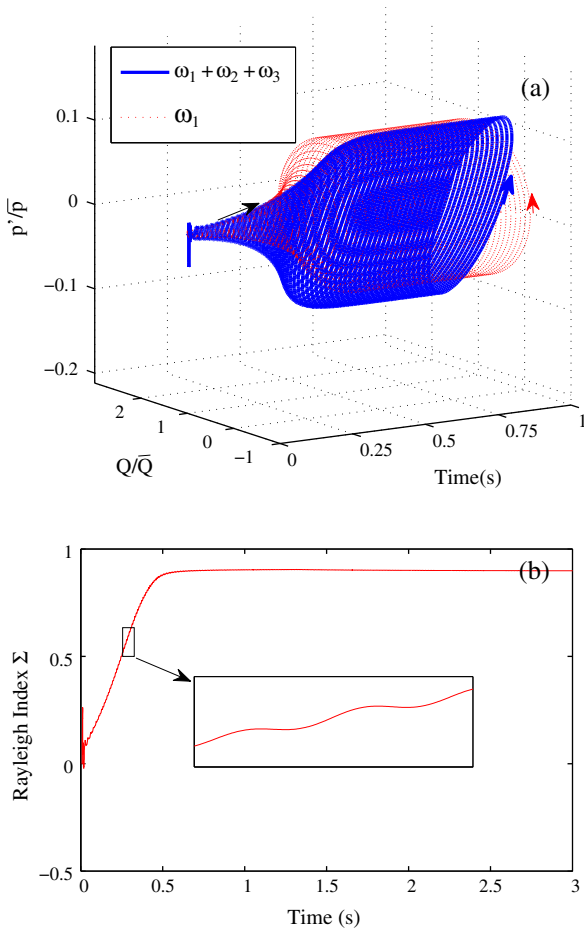


Fig. 4. Time evolution of (a) the phase diagram between $Q(t)/\bar{Q}$ and $p'(t)/\bar{p}$; (b) Rayleigh index $\Sigma(t)$.

oscillating between -180° and 180° . The decrease of the magnitude of T_r with increased S_t indicates that the flame is more sensitive to respond to lower frequency acoustic disturbances. However, high frequency acoustic disturbances pass through the flame more smoothly.

The nonlinear response of the flame to the oncoming flow disturbances is then determined by numerically solving Eqs. (13), (16) and (34) with specified conditions as given in Table A.1. Fig. 4(a) shows the time evolution of the phase diagram between unsteady heat release $Q(t)/\bar{Q}$ and pressure fluctuation $p'(t)/\bar{p}$, as denoted by the solid line. It can be seen that limit cycle is generated at $t = 0.5$ s. The dash line describes the phase diagram between $Q(t)/\bar{Q}$ and $p'(t)/\bar{p}$, which is assumed to consist of only the fundamental mode at ω_1 . When the phase difference between them is $\pi/2$, circular-shape is observed. The mode growth rate is approximately $e^{6.015t}$.

Fig. 4(b) shows the variation of the Rayleigh index $\Sigma(t)$ with time. It can be seen that when the initial perturbation grows into a limit cycle, Rayleigh index Σ is positive, keeps increased and finally 'saturated' at approximately 0.90. This verifies our numerical model and explains why self-sustained combustion oscillations (i.e. thermoacoustic instability) occur.

In order to quantify the eigenmodes nonorthogonality, the eigenfrequencies ω_m need to be determined by using Eq. (44). Fig. 5 shows the variation of estimated ω_1 with the flame location z_f/L . Compared with the experimental measurements [29], qualitatively good agreement is observed. It is worth noting that the measured frequency at fixed flame location is changed with varied fuel–air ratio.

With ω_m estimated, the inner product $\langle \hat{p}_m, \hat{p}_n^* \rangle$ characterizing eigenmodes nonorthogonality can be calculated, by assuming that $\hat{p}_u^- = 1.0 + j0$, $\hat{p}_u^+ = \hat{Z}_u \hat{p}_u^- \exp\{-j\omega\tau_u\}$, $\hat{p}_d^+ = -\hat{p}_u^- \mathbf{W}_{11}/\mathbf{W}_{12}$, and $\hat{p}_d^- = \hat{Z}_d \hat{p}_d^+ \exp\{-j\omega\tau_d\}$. Fig. 6(a) illustrates the variation of the inner product $\langle \hat{p}_m, \hat{p}_n^* \rangle$ with the flame position z_f/L . It is apparent that the thermoacoustic modes are nonorthogonal, since $\|\langle \hat{p}_m, \hat{p}_n^* \rangle\| > 0$. Furthermore, the nonorthogonality depends on the flame location z_f/L . It is also observed that the nonorthogonality is increased with increased flame-acoustic interaction index $\mathcal{N}$, as shown in Fig. 6(b). This indicates that the system nonnormality becomes more pronounced with the interaction intensity between the flame and oncoming acoustic disturbances.

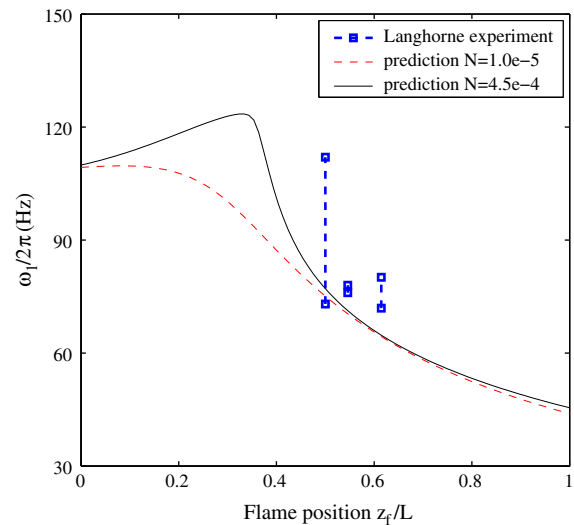


Fig. 5. Comparison between the predicted mode frequency ω_1 and the experimentally measured ones, as $L = 1.92$ m, $\bar{M}_u = 0.08$, $\tau_u = 0.01$ s: - - - $\mathcal{N} = 1.0e-5$, — $\mathcal{N} = 4.5e-4$.

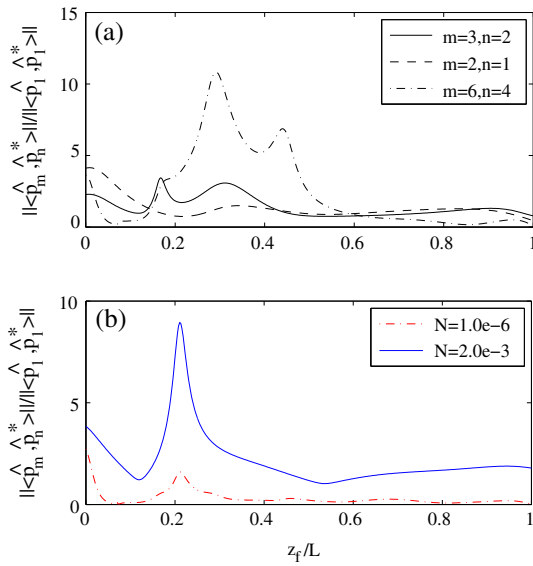


Fig. 6. The variation of $\|\langle \hat{p}_m, \hat{p}_n \rangle\| / \|\langle \hat{p}_1, \hat{p}_1 \rangle\|$ with the flame location z_f/L , as $\tau_v = 0.01$ s, $L = 1.92$ m, and $\bar{M}_u = 0.08$, $\bar{c}_u = 340$ m/m and $T_u = 288$ K.

For the combustor as modeled in Section 2, the inlet is choked and so the mass flow rate is constant at $z = z_u$, i.e. $\bar{p}u' + \bar{u}\rho' = 0$. Further analysis shows that the acoustic impedance at this choked inlet $\hat{Z}(z_u) = \hat{p}/\hat{u} = -\gamma\bar{p}/\bar{u}_u \neq 0$ and the boundary condition is a Robin-type one, i.e. $\hat{Z}\partial\hat{p}/\partial z + j\omega\bar{p}_u\hat{p} = 0$. It is a weighted combination of Dirichlet- and Neumann-type boundary condition [26]. At the open outlet (corresponding to Dirichlet-type boundary condition), the pressure fluctuation is zero, i.e. $\hat{p}(z_d) = 0$ and so the acoustic impedance $\hat{Z}_d = 0$. The estimated inner product is shown in the bar chart as in Fig. 7. It can be seen that the acoustic modes are nonorthogonal. However, the order of magnitude of the inner product is much smaller than that in the presence of the flame. This indicates that the complex boundary condition of the thermoacoustic system does contribute to the eigenmodes nonorthogonality. However, the contribution is approximately 1.5% of that from the heat source–flame. In practice, there are many combustion systems such as baffled liquid-propellant rocket engines [1] or swirl-stabilized gas turbines, which involve with such choked boundary condition. These combustion systems are associated with nonorthogonal eigenmodes, whether or not a flame is present. This is different from those combustors with Dirichlet- or Neumann-type boundary conditions as analyzed in Appendix C;

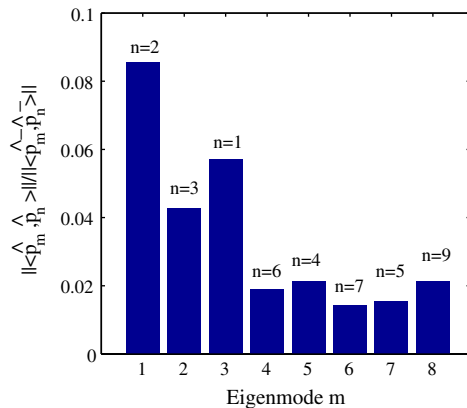


Fig. 7. The variation of $\|\langle \hat{p}_m, \hat{p}_n \rangle\|$ normalized with $\|\langle \hat{p}_m, \hat{p}_m \rangle\|$ with m and n , as $L = 1.92$ m and $\bar{M}_u = 0.08$.

they are associated with orthogonal eigenmodes in the absence of the heat source.

In general, eigenmodes nonorthogonality analysis reveals that the choked combustion system with the V-shaped premixed flame confined is a nonnormal thermoacoustic system. The nonnormality characterized by nonorthogonal eigenmodes arises from both the complex boundary condition and the flame. However, the contribution to the nonorthogonality from the heat source is identified to play a dominant role. The nonorthogonality is also found to depend on the flame location. In such a nonnormal system, small acoustic disturbances may exhibit transient growth before they eventually decay. If this transient growth is large enough, it can trigger combustion instabilities [31–46].

5. Conclusions

In this work, nonorthogonality analysis of a combustion system with a choked inlet and open exit is conducted. For this, a thermoacoustic model is developed to study the nonnormal interaction between a premixed V-shaped turbulent flame and acoustic disturbances. The state variables are expanded in terms of eigenmodes determined as the solutions of the linearized momentum and energy equations. It is shown the choked combustion system is nonnormal and characterized by nonorthogonal eigenmodes. The nonorthogonality is identified to arise from both the complex boundary condition and the heat source–flame. However, the contribution from the Robin-type boundary is approximately 1.5% of that from the flame. This indicates that the flame plays an important role in contributing to the system nonorthogonality. This is different from the combustion systems with Dirichlet- or Neumann-type boundary conditions, of which the nonorthogonality results mainly from the heat source. Such boundaries do not contribute to the system nonorthogonality. The eigenmode nonorthogonality results in the transient (finite-time) growth of acoustic disturbances, which may have great potential to trigger thermoacoustic instability. Classical linear stability theory is not enough to predict such finite-time system stability behaviors. Thus understanding and estimating the system nonorthogonality are important.

Acknowledgements

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Appendix A. Geometry of the combustor

See Table A.1.

Table A.1
Geometry and flow parameters of the combustion system.

Geometry	Value
Combustor length, L (m)	1.92
Radius of the combustor, R_b (m)	0.035
Radius of the flame-holder/gutter, R_a (m)	0.0175
Upstream length of the combustor, $ z_f - z_u $ (m)	1.18
Downstream length of the combustor $ z_d - z_f $, (m)	0.74
Combustor inlet Mach number, $\bar{M}_u$	0.08
Combustor inlet temperature, T_u (K)	288
Combustor inlet and outlet pressure, $\bar{p}_u = \bar{p}_d$ (Pa)	1.01325×10^5
Fuel,	Ethylene
Mean heat release rate $\bar{Q}_f$, (J)	3.16×10^6
Fuel–air ratio normalized on stoichiometric fuel–air ratio,	0.7
Flame speed (assumed constant) (m/s)	3.6
Combustion efficiency	0.8
Speed of sound at combustor inlet (m/s)	340.17

Appendix B. Coefficient matrices

The coefficient matrices **A** and **B** are given as

$$\mathbf{A} = \begin{pmatrix} -1 + \overline{M}_u \left(2 - \frac{\bar{u}_d}{u_d} \right) - \overline{M}_u^2 & 1 + \overline{M}_u \frac{\bar{\rho}_u \bar{c}_u}{\bar{\rho}_d \bar{c}_d} \\ \frac{1-\gamma}{\gamma-1} \overline{M}_u + \overline{M}_u^2 - \frac{\overline{M}_u^2}{2} (1 - \overline{M}_u) \left(\frac{\bar{u}_d^2}{u_d^2} - 1 \right) & \overline{M}_u \overline{M}_d \frac{\bar{\rho}_u}{\bar{\rho}_d} + \frac{\bar{c}_d}{\bar{c}_u} \frac{1+\gamma \overline{M}_d}{\gamma-1} \end{pmatrix} \quad (\text{B.1})$$

$$\mathbf{B} = \begin{pmatrix} \frac{1-\overline{M}_u}{1+\overline{M}_u} \left(1 + \overline{M}_u \left(2 - \frac{\bar{u}_d}{u_d} \right) + \overline{M}_u^2 \right) & 1 - \overline{M}_u \frac{\bar{\rho}_u \bar{c}_u}{\bar{\rho}_d \bar{c}_d} \\ \frac{1+\gamma \overline{M}_u}{\gamma-1} + \overline{M}_u^2 - \frac{\overline{M}_u^2}{2} (1 + \overline{M}_u) \left(\frac{\bar{u}_d^2}{u_d^2} - 1 \right) & -\frac{\bar{c}_d}{\bar{c}_u} \frac{1-\gamma \overline{M}_d}{\gamma-1} - \overline{M}_u \overline{M}_d \frac{\bar{\rho}_u}{\bar{\rho}_d} \end{pmatrix} \quad (\text{B.2})$$

Appendix C. Nonorthogonality analysis of combustors with Dirichlet- and Neumann-type boundary conditions

Combustors are associated with two general types of boundary conditions, which are widely used in numerical and theoretical analysis. They are namely Dirichlet- or Neumann-type. In order to analyze the eigenmodes orthogonality of the combustors with such boundary conditions, Eq. (50) can be used:

- (1) Dirichlet (corresponding to open end), i.e. $\hat{p}(z_u) = \hat{p}(z_d) = 0$. The inner product $\langle \hat{p}_m, \hat{p}_n \rangle$ as indicator of orthogonality can be shown as

$$\langle \hat{p}_m, \hat{p}_n \rangle = \int_{z_u}^{z_d} \hat{p}_m \hat{p}_n dz = \frac{\bar{c}^2}{(\omega_n^2 - \omega_m^2)} \left[\frac{\partial \hat{p}_m}{\partial z} \hat{p}_n - \frac{\partial \hat{p}_n}{\partial z} \hat{p}_m \right]_{z_u}^{z_d} = 0 \quad (\text{C.1})$$

It can be seen that the combustors with the Dirichlet-type boundary conditions are associated with orthogonal acoustic modes. The acoustic impedance $Z = p'/u'$ is zero at the open end in the absence of heat source. However, when a heat source is present, the combustion systems become nonnormal characterized with nonorthogonal eigenmodes [25]. This indicates that nonnormality arises from the heat source, instead of the boundary conditions.

- (2) Neumann-type boundary condition, i.e. $u'(z_u, t) = 0$ and $u'(z_d, t) = 0$.

Analyzing the nonorthogonality of combustors with Neumann-type boundary conditions begins with estimating the inner product of the eigenmodes $\langle \hat{p}_m, \hat{p}_n \rangle$ as given in Eq. (C.2). It can be shown that $\langle \hat{p}_m, \hat{p}_n \rangle$ is zero. This means that orthogonal modes are involved and the system is normal. The acoustic impedance at the close end is infinitely large $Z \rightarrow \infty$.

$$\begin{aligned} \langle \hat{p}_m, \hat{p}_n \rangle &= \int_{z_u}^{z_d} \hat{p}_m \hat{p}_n dz \\ &= \frac{\bar{c}^2}{(\omega_n^2 - \omega_m^2)} \left[\frac{\partial \hat{p}_m}{\partial z} \hat{p}_n - \frac{\partial \hat{p}_n}{\partial z} \hat{p}_m \right]_{z_u}^{z_d} \\ &= \frac{\bar{\rho} \bar{c}^2}{(\omega_n^2 - \omega_m^2)} \left[\frac{\partial \hat{u}_m}{\partial t} \hat{p}_n - \frac{\partial \hat{u}_n}{\partial t} \hat{p}_m \right]_{z_u}^{z_d} = 0 \end{aligned} \quad (\text{C.2})$$

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